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A new solution of the Yang–Baxter equation related to the adjoint representation of U_qB_2

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Abstract. A new solution of the Yang–Baxter equation, that is related to the adjoint representation of the quantum enveloping algebra U_qB_2, is obtained by fusion formulae from a non-standard solution.

1. Introduction

There are three typical methods [1] for finding the trigonometric solutions of the Yang–Baxter equation [2]. The main one is based on Jimbo's theorem [3,4]. The necessary condition for using this method is the existence of the quantum generator e_0, corresponding to the negative lowest root. The second method for finding solutions is the so-called Yang–Baxterization, namely to embed appropriately the spectral parameter x into a solution $\breve{R}_q$ of the simple Yang–Baxter equation such that $\breve{R}_q(x)$ satisfies the Yang–Baxter equation. This method is useful for the cases where the spectrum-independent solution $\breve{R}_q$ has only two or three different eigenvalues [5,1]. The third method is the fusion formulae [6,7,1] where an appropriate project operator is needed.

Unfortunately, firstly, the explicit form of e_0, satisfying the quantum algebraic relations, does not exist for the adjoint representation of any quantum enveloping algebra $U_q\mathcal{G}$, except for U_qA_ℓ; however, for the quantum twisted loop algebra, as reminded by the referee, the e_0 matrix may exist. Secondly, the spectrum-independent solution $\breve{R}_q$ for the adjoint representation usually has many more than three different eigenvalues. For instance, in the simplest case, the solution $\breve{R}_q$ for the adjoint representation of U_qB_2 has six different eigenvalues. Finally, from the solution $\breve{R}_q(x)$ related to the minimal representation, obtained based on Jimbo's theorem, the needed project operator for the fusion formulae does not exist for this case. This is the reason why no solution of the Yang–Baxter equation related to the adjoint representation of $U_q\mathcal{G}$, except for U_qA_ℓ, has been found up to now.

On the other hand, by Yang–Baxterization, when $\breve{R}_q$ has three different eigenvalues, there is an additional solution, a so-called non-standard one, that happens to provide the needed project operator for the fusion formulae. In this way we are able to compute the solution related to the adjoint representations of U_qB_ℓ, U_qC_ℓ and U_qD_ℓ. In order to realize this idea, in this paper we compute explicitly the simplest example of those cases: the trigonometric and rational solutions of the Yang–Baxter equation related to the adjoint representation of U_qB_2, that is equivalent to U_qC_2. The rest of the solutions can be computed straightforwardly, but more complicatedly.

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This paper is organized as follows. In section 2, we show that the explicit form of the e_0 matrix for the adjoint representation of U_qB_2, that satisfies the quantum algebraic relations, does not exist. In order to use the fusion formulae, firstly we have to compute the solution $\breve{R}_q(x)$ of the Yang–Baxter equation related to the minimal representation in section 3. From this we obtain the project operator $\breve{R}_q(q^{-4})$ that maps the direct product spaces $V_{(1\,0)} \otimes V_{(1\,0)}$ onto the representation space $V_{\rm adj} = V_{(0\,2)}$ of the adjoint representation, where $V_{(1\,0)}$ are the representation spaces of the minimal representation (1 0). In section 4 we sketch the proof for the fusion formulae. The explicit form of $\breve{R}_q^{\rm adj}(x)$ is computed in section 5 in terms of the quantum Clebsch–Gordan coefficients for the co-product in the direct product of two representation spaces of the adjoint representation. The corresponding rational solution of the Yang–Baxter equation is obtained in section 6 by a standard limit process [1].

2. Non-existence of e_0 matrix

The Cartan matrix for the algebra B_2 is

$$a = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \qquad a^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}. \tag{1}$$

From this we have the relation between the simple roots r_j and the fundamental dominant weight λ_j:

$$r_1 = 2\lambda_1 - 2\lambda_2 \qquad r_2 = -\lambda_1 + 2\lambda_2 \qquad \lambda_1 = r_1 + r_2 \qquad \lambda_2 = r_1/2 + r_2. \tag{2}$$

An irreducible representation of U_qB_2 is denoted by its highest weight $\boldsymbol{M} = (M_1M_2)$ and the states by $\boldsymbol{m} = (m_1m_2)$:

$$\boldsymbol{M} = M_1\lambda_1 + M_2\lambda_2 \qquad \boldsymbol{m} = m_1\lambda_1 + m_2\lambda_2. \tag{3}$$

The minimal representation is denoted by (1 0), and the adjoint representation by (0 2). The Casimir $C_2(\boldsymbol{M})$ is calculated by the following formula:

$$C_2(\boldsymbol{M}) = M_1^2 + M_1M_2 + M_2^2/2 + 3M_1 + 2M_2. \tag{4}$$

The quantum algebraic relations for U_qB_2 are as follows:

$$\begin{aligned} & q_1 = q_2^2 = q \qquad k_ik_j = k_jk_i \qquad k_ie_j = q_i^{a_{ij}}e_jk_i \qquad k_if_j = q_i^{-a_{ij}}f_jk_i \\ & [e_i, f_j] = \delta_{ij}\frac{k_j^2 - k_j^{-2}}{q_j^2 - q_j^{-2}} \qquad i, j = 1, 2 \\ & \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{pmatrix} 1 - a_{ij} \\ n \end{pmatrix}_{q_j^2} e_i^{1-a_{ij}-n} e_j e_j^n = 0 \qquad i \neq j \\ & \sum_{n=0}^{1-a_{ij}} (-1)^n \begin{pmatrix} 1 - a_{ij} \\ n \end{pmatrix}_{q_j^2} f_i^{1-a_{ij}-n} f_j f_j^n = 0 \qquad i \neq j. \end{aligned} \tag{5}$$

From Jimbo's theorem [3], we hope to find the generators k_0, e_0 and f_0, corresponding to the negative lowest root r_0 of B_2

$$r_0 = -2\lambda_2 = -r_1 - 2r_2 \tag{6}$$

such that they satisfy the quantum algebraic relations (5) with $i, j = 0, 1,$ and 2, where

$$k_0 = k_1^{-1}k_2^{-2} \qquad a_{00} = 2 \qquad a_{01} = a_{10} = 0 \qquad a_{02} = -1 \qquad a_{20} = -2\,. \tag{5$'$}$$

In the following we are going to show that for the adjoint representation of $U_q B_2$ those quantum representation matrices satisfying (5) do not exist.

Through a standard method [1] we draw the block weight diagrams for the representations (1 0) and (0 2) in figure 1.

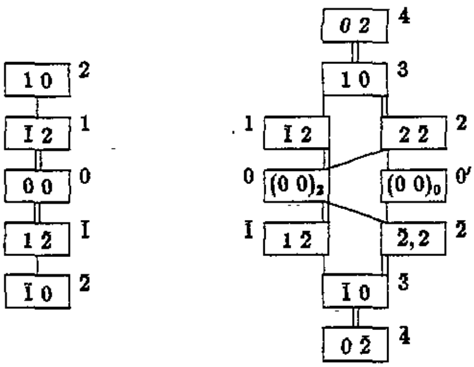

Figure 1. Block weight diagrams for (*a*) the minimal and (*b*) the adjoint representations of algebra $U_q B_2$

In order to simplify the notation we enumerate the states in those two representations as shown near the blocks in figure 1. In terms of the enumerations for the states and the matrix bases E_{ab}

$$(E_{ab})_{cd} = \delta_{ac}\delta_{bd} \tag{7}$$

we obtain the quantum representation matrices for two representations as follows. For the minimal representation (1 0) we have

$$\begin{aligned}
D_q(e_1) &= \tilde{D}_q(f_1) = E_{21} + E_{\bar{1}\bar{2}} \\
D_q(e_2) &= \tilde{D}_q(f_2) = [2]^{1/2}\left(E_{10} + E_{0\bar{1}}\right) \\
D_q(k_1) &= qE_{22} + q^{-1}E_{11} + E_{00} + qE_{\bar{1}\bar{1}} + q^{-1}E_{\bar{2}\bar{2}} \\
D_q(k_2) &= E_{22} + qE_{11} + E_{00} + q^{-1}E_{\bar{1}\bar{1}} + E_{\bar{2}\bar{2}}
\end{aligned} \tag{8}$$

and for the adjoint representation (0 2) we have

$$\begin{aligned}
D_q(e_1) &= \tilde{D}_q(f_1) = E_{31} + E_{20} + \left(\frac{[6]}{[3][2]}\right)^{1/2} \left(E_{20'} + E_{0'\bar{2}}\right) + E_{0\bar{2}} + E_{\bar{1}\bar{3}} \\
D_q(e_2) &= \tilde{D}_q(f_2) = [2]^{1/2} \left(E_{43} + E_{32} + E_{10} + E_{0\bar{1}} + E_{\bar{2}\bar{3}} + E_{\bar{3}\bar{4}}\right) \\
D_q(k_1) &= E_{44} + qE_{33} + q^2 E_{22} + q^{-1} E_{11} + E_{00} + E_{0'0'} + qE_{\bar{1}\bar{1}} \\
&\qquad + q^{-2} E_{\bar{2}\bar{2}} + q^{-1} E_{\bar{3}\bar{3}} + E_{\bar{4}\bar{4}} \\
D_q(k_2) &= qE_{44} + E_{33} + q^{-1} E_{22} + qE_{11} + E_{00} + E_{0'0'} + q^{-1} E_{\bar{1}\bar{1}} \\
&\qquad + qE_{\bar{2}\bar{2}} + E_{\bar{3}\bar{3}} + q^{-1} E_{\bar{4}\bar{4}}
\end{aligned} \tag{9}$$

where the tilde denotes transpose, and, as usual, $[m]$ denotes

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{10}$$

Owing to (6), the possible forms of the representation matrices of e_0 and f_0, that correspond to r_0, are as follows:

$$\begin{aligned}
D_q(e_0) &= a_1 E_{04} + a_2 E_{0'4} + a_3 E_{\bar{1}3} + a_4 E_{\bar{3}1} + a_5 E_{\bar{4}0} + a_6 E_{\bar{4}0'} \\
D_q(f_0) &= b_1 E_{40} + b_2 E_{40'} + b_3 E_{3\bar{1}} + b_4 E_{1\bar{3}} + b_5 E_{0\bar{4}} + b_6 E_{0'\bar{4}}.
\end{aligned} \tag{11}$$

From the quantum algebraic relations

$$[D_q(e_0), D_q(f_j)] = 0 \qquad [D_q(f_0), D_q(e_j)] = 0 \qquad j = 1, 2 \tag{12}$$

we obtain

$$\begin{aligned}
-\left(\frac{[6]}{[3][2]}\right)^{1/2} a_2 &= -\left(\frac{[6]}{[3][2]}\right)^{1/2} a_6 = a_1 = a_3 = a_4 = a_5 \\
-\left(\frac{[6]}{[3][2]}\right)^{1/2} b_2 &= -\left(\frac{[6]}{[3][2]}\right)^{1/2} b_6 = b_1 = b_3 = b_4 = b_5.
\end{aligned} \tag{13}$$

It is easy to check that the quantum Serre relations are not satisfied:

$$\begin{aligned}
&D_q(e_0)^2 D_q(e_2) - ([4]/[2])\, D_q(e_0) D_q(e_2) D_q(e_0) + D_q(e_2) D_q(e_0)^2 \\
&\qquad = -\left(q^{-1} - q\right)^2 \left(\frac{[4][3]}{[6]}\right) a_1^2 [2]^{1/2} \left(E_{\bar{4}3} + E_{\bar{3}4}\right) \neq 0 \\
&D_q(f_0)^2 D_q(f_2) - ([4]/[2])\, D_q(f_0) D_q(f_2) D_q(f_0) + D_q(f_2) D_q(f_0)^2 \\
&\qquad = -\left(q^{-1} - q\right)^2 \left(\frac{[4][3]}{[6]}\right) b_1^2 [2]^{1/2} \left(E_{3\bar{4}} + E_{4\bar{3}}\right) \neq 0.
\end{aligned} \tag{14}$$

The commutator of $D_q(e_0)$ and $D_q(f_0)$ does not satisfy the quantum algebraic relations (5), either. Therefore, the representation matrix $D_q(e_0)$ does not exist for the adjoint representation of $U_q B_2$.

3. Solutions for the minimal representation

In the fusion formulae, the $\check{R}_q^{\mathrm{adj}}(x)$ matrix for the adjoint representation is expressed in terms of the $\check{R}_q(x)$ matrix for the minimal representation. In this section we compute the $\check{R}_q(x)$ matrix for the minimal representation firstly. As a matter of fact, the e_0 matrix exists in the minimal representation of U_qB_2 so that the corresponding solution $\check{R}_q(x)$ was computed [3,1] by the standard method based on Jimbo's theorem.

As usual, the co-product of the generators in two irreducible representations

$$\Delta_q^{M_1M_2}(k_j) = D_q^{M_1}(k_j) \otimes D_q^{M_2}(k_j)$$

$$\Delta_q^{M_1M_2}(e_j) = D_q^{M_1}(e_j) \otimes D_q^{M_2}(k_j^{-1}) + D_q^{M_1}(k_j) \otimes D_q^{M_2}(e_j)$$

$$\Delta_q^{M_1M_2}(f_j) = D_q^{M_1}(f_j) \otimes D_q^{M_2}(k_j^{-1}) + D_q^{M_1}(k_j) \otimes D_q^{M_2}(f_j)$$

is a reducible representation of the quantum enveloping algebra, and can be reduced by the quantum Clebsch–Gordan matrix

$$\Delta_q^{M_1M_2}(I)(C_q^{M_1M_2})_N = (C_q^{M_1M_2})_N D_q^N(I) \qquad I = k_j, e_j \text{ and } f_j\,.$$

The Clebsch–Gordan series for the direct product of two minimal representations is as follows:

$$(1\,0) \otimes (1\,0) = (2\,0) \oplus (0\,2) \oplus (0\,0)\,. \tag{15}$$

$\mathcal{P}_N$ denotes the project operator that is the product of two quantum Clebsch–Gordan matrices [1]

$$\mathcal{P}_N = \left(C_q\right)_N \left(\tilde{C}_q\right)_N \tag{16}$$

where the superscripts of the Clebsch–Gordan matrix for the minimal representation are omitted for simplicity. By making use of the standard method based on Jimbo's theorem, we obtain the $\check{R}'_q(x)$ matrix for the minimal representation as follows [3,1]:

$$\check{R}'_q(x) = (1-xq^4)(1-xq^6)\mathcal{P}_{(20)} + (x-q^4)(1-xq^6)\mathcal{P}_{(02)} + (x-q^4)(x-q^6)\mathcal{P}_{(00)} \tag{17}$$

where a prime is added on $\check{R}'_q(x)$ in order to distinguish it from the additional solution $\check{R}_q(x)$ given in (18). In the form of $\check{R}'_q(x)$, it cannot be proportional to the projector operator $\mathcal{P}_{(0\,2)}$ that maps the direct product space onto the space of the adjoint representation. In the same paper [3] Jimbo pointed out that there is another solution related to the algebra $U_qA_4^{(2)}$, namely

$$\check{R}_q(x) = (1-xq^4)(1+xq^{10})\mathcal{P}_{(20)} + (x-q^4)(1+xq^{10})\mathcal{P}_{(02)} + (1-xq^4)(x+q^{10})\mathcal{P}_{(00)}\,. \tag{18}$$

Now, we know [5,1] that because the Clebsch–Gordan series (15) contains only three representations including an identity representation (00), we can obtain two independent solutions of the Yang–Baxter equation given in (17) and (18) in terms of Yang–Baxterization. The solution (18), called non-standard, has a good property,

$$\check{R}_q(q^{-4}) = (q^{-4}-q^4)(1+q^6)\mathcal{P}_{(0\,2)} \tag{19}$$

namely, $\check{R}_q(q^{-4})$ is proportional to the project operator $\mathcal{P}_{(02)}$ onto the adjoint representation

$$\check{R}_q(q^{-4})(V_{(10)} \otimes V_{(10)}) = V_{(02)} \,. \tag{20}$$

It is the key point for computing the solution related to the adjoint representation from (18). Since the solution (18), as pointed out by Jimbo, is related to the algebra $U_q A_4^{(2)}$, the representation matrix of e_0 satisfying the quantum algebraic relation of $U_q A_4^{(2)}$ may exist. The author would like to thank the referee for reminding us about this point. We will discuss this problem elsewhere.

Solution (18) is a 25×25 symmetric matrix on the direct product space $V_{(10)} \otimes V_{(10)}$. The row (column) indices are denoted by $m_1 m_2$, where both m_1 and m_2 take the values $2, 1, 0, \bar{1}$, and $\bar{2}$. $\check{R}_q(x)$ has the following symmetries:

$$\begin{gathered}\check{R}_q(x)_{m_1m_2\ m_3m_4} = \check{R}_q(x)_{m_3m_4\ m_1m_2} = \check{R}_q(x)_{\bar{m}_2\bar{m}_1\ \bar{m}_4\bar{m}_3} = -x^2 q^{14} \check{R}_{q^{-1}}(x^{-1})_{m_2m_1\ m_4m_3} \\ \check{R}_q(1)_{m_1m_2\ m_3m_4} = (1-q^4)(1+q^{10})\delta_{m_1m_3}\delta_{m_2m_4}\end{gathered} \tag{21}$$

where $\bar{0} = 0$.

$\check{R}_q(x)$ given in (18) satisfies the weight conservation condition, namely, $\check{R}_q(x)$ is a block matrix with four 1×1, eight 2×2 and one 5×5 submatrices. Through straightforward calculation we obtain the explicit form for $\check{R}_q(x)$. Owing to the symmetries (21) we only need to list the results as follows:

(a) four 1×1 submatrices:

$$\check{R}_q(x)_{22\,22} = \check{R}_q(x)_{11\,11} = (1-xq^4)(1+xq^{10}) \tag{22a}$$

(b) eight 2×2 submatrices:

$$\begin{aligned}\check{R}_q(x)_{21\,21} = \check{R}_q(x)_{20\,20} = \check{R}_q(x)_{2\bar{1}\,2\bar{1}} = \check{R}_q(x)_{10\,10} = (1-q^4)x(1+xq^{10}) \\ \check{R}_q(x)_{21\,12} = \check{R}_q(x)_{20\,02} = \check{R}_q(x)_{2\bar{1}\,\bar{1}2} = \check{R}_q(x)_{10\,01} = q^2(1-x)(1+xq^{10})\end{aligned} \tag{22b}$$

(c) one 5×5 submatrix:

$$\begin{aligned}
\check{R}_q(x)_{2\bar{2}\,2\bar{2}} &= (1-q^4)x\left\{(1+q^4) - xq^4(1-q^6)\right\} \\
\check{R}_q(x)_{1\bar{1}\,1\bar{1}} &= (1-q^4)x\left\{(1+q^8) - xq^8(1-q^2)\right\} \\
\check{R}_q(x)_{00\,00} &= q^2(1-x)(1+xq^{10}) + x(1-q^4)(1+q^{10}) \\
\check{R}_q(x)_{2\bar{2}\,1\bar{1}} &= -x(1-x)q^6(1-q^4) \\
\check{R}_q(x)_{2\bar{2}\,00} &= x(1-x)q^7(1-q^4) \\
\check{R}_q(x)_{2\bar{2}\,\bar{1}1} &= -x(1-x)q^8(1-q^4) \\
\check{R}_q(x)_{1\bar{1}\,00} &= -x(1-x)q^9(1-q^4) \\
\check{R}_q(x)_{2\bar{2}\,\bar{2}2} &= \check{R}_q(x)_{1\bar{1}\,\bar{1}1} = q^4(1-x)(1+xq^6)
\end{aligned} \tag{22c}$$

4. Fusion formulae

The project operator $\check{R}_q(q^{-4})$ maps the direct product space $V_{(10)} \otimes V_{(10)}$ of two minimal representations onto the representation space $V_{(02)}$ of the adjoint representation. The solution $\check{R}_q^{\text{adj}}(x)$ of the Yang–Baxter equation related to the adjoint representation of $U_q B_2$ is applied on the direct product space $V_{(02)} \otimes V_{(02)}$

$$V_{(02)} \otimes V_{(02)} = \left(\check{R}_q(q^{-4}) \otimes \check{R}_q(q^{-4})\right)\left(V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\right) . \tag{23}$$

According to the fusion formulae, $\check{R}_q^{\text{adj}}(x)$ can be expressed as the following product [6,1]:

$$\check{R}_q^{\text{adj}}(x) = \big(\mathbf{1} \otimes \check{R}_q(xq^4) \otimes \mathbf{1}\big)\big(\check{R}_q(x) \otimes \check{R}_q(x)\big)\big(\mathbf{1} \otimes \check{R}_q(xq^{-4}) \otimes \mathbf{1}\big). \tag{24}$$

Now, we are going to sketch the proof. First of all, we show that $\check{R}_q^{\text{adj}}(x)$ given in (24) is a matrix on the space (23). From the Yang–Baxter equation satisfied by $\check{R}_q(x)$:

$$\big(\mathbf{1} \otimes \check{R}_q(x)\big)\big(\check{R}_q(xy) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \check{R}_q(y)\big) = \big(\check{R}_q(y) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \check{R}_q(xy)\big)\big(\check{R}_q(x) \otimes \mathbf{1}\big) \tag{25}$$

we have

$$\begin{aligned}
\check{R}_q^{\text{adj}}(x)\big(V_{(02)} \otimes V_{(02)}\big) &= \big(\mathbf{1} \otimes \check{R}_q(xq^4) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(x)\big) \\
&\quad \times \big(\check{R}_q(x) \otimes \mathbf{1} \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \check{R}_q(xq^{-4}) \otimes \mathbf{1}\big)\big(\check{R}_q(q^{-4}) \otimes \mathbf{1} \otimes \mathbf{1}\big) \\
&\quad \times \big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big)\big\{V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big\} \\
&= \big(\mathbf{1} \otimes \check{R}_q(xq^4) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(x)\big)\big(\mathbf{1} \otimes \check{R}_q(q^{-4}) \otimes \mathbf{1}\big) \\
&\quad \times \big(\check{R}_q(xq^{-4}) \otimes \mathbf{1} \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \check{R}_q(x) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big) \\
&\quad \times \big(V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big) \\
&= \big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big)\big(\mathbf{1} \otimes \check{R}_q(x) \otimes \mathbf{1}\big)\big(\check{R}_q(xq^{-4}) \otimes \mathbf{1} \otimes \mathbf{1}\big) \\
&\quad \times \big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(xq^4)\big)\big(\mathbf{1} \otimes \check{R}_q(x) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big) \\
&\quad \times \big(V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big) \\
&= \big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big)\big(\mathbf{1} \otimes \check{R}_q(x) \otimes \mathbf{1}\big)\big(\check{R}_q(xq^{-4}) \otimes \mathbf{1} \otimes \mathbf{1}\big) \\
&\quad \times \big(\mathbf{1} \otimes \check{R}_q(q^{-4}) \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(x)\big)\big(\mathbf{1} \otimes \check{R}_q(xq^4) \otimes \mathbf{1}\big) \\
&\quad \times \big(V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big) \\
&= \big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(q^{-4})\big)\big(\check{R}_q(q^{-4}) \otimes \mathbf{1} \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \check{R}_q(xq^{-4}) \otimes \mathbf{1}\big) \\
&\quad \times \big(\check{R}_q(x) \otimes \mathbf{1} \otimes \mathbf{1}\big)\big(\mathbf{1} \otimes \mathbf{1} \otimes \check{R}_q(x)\big)\big(\mathbf{1} \otimes \check{R}_q(xq^4) \otimes \mathbf{1}\big) \\
&\quad \times \big(V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big) \\
&\quad \subset \big(\check{R}_q(q^{-4}) \otimes \check{R}_q(q^{-4})\big)\big\{V_{(10)} \otimes V_{(10)} \otimes V_{(10)} \otimes V_{(10)}\big\} \\
&= V_{(02)} \otimes V_{(02)} .
\end{aligned}$$

By making use of (25) successively, it is straightforward to prove that $\check{R}_q^{\rm adj}(x)$ satisfies the Yang–Baxter equation, that is an equation on the direct product space $V_{(10)}^{\otimes 6}$:

$$\begin{aligned}
&\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(xq^4)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(x)\otimes\check{R}_q(x)\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(xq^{-4})\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\check{R}_q(xyq^4)\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big)\big(\check{R}_q(xy)\otimes\check{R}_q(xy)\otimes\mathbf{1}\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\check{R}_q(xyq^{-4})\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(yq^4)\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(y)\otimes\check{R}_q(y)\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(yq^{-4})\otimes\mathbf{1}\big)\\
&\quad=\big(\mathbf{1}\otimes\check{R}_q(yq^4)\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big)\big(\check{R}_q(y)\otimes\check{R}_q(y)\otimes\mathbf{1}\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\check{R}_q(yq^{-4})\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(xyq^4)\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(xy)\otimes\check{R}_q(xy)\big)\big(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\check{R}_q(xyq^{-4})\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\check{R}_q(xq^4)\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big)\big(\check{R}_q(x)\otimes\check{R}_q(x)\otimes\mathbf{1}\otimes\mathbf{1}\big)\\
&\qquad\times\big(\mathbf{1}\otimes\check{R}_q(xq^{-4})\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\big).
\end{aligned}\tag{26}$$

5. Explicit form of the solution for the adjoint representation

The Clebsch–Gordan series for the direct product of two adjoint representations of B_2 is

$$(0\,2)\otimes(0\,2)=(0\,4)\oplus(1\,2)\oplus(2\,0)\oplus(0\,2)\oplus(1\,0)\oplus(0\,0)\,. \tag{27}$$

Both the solution $\check{R}_q^{\rm adj}$ of the simple Yang–Baxter equation and the solution $\check{R}_q^{\rm adj}(x)$ of the Yang–Baxter equation, related to the adjoint representation of U_qB_2, can be expanded by the project operators as follows:

$$\check{R}_q^{\rm adj}=\mathcal{P}_{(04)}-q^4\mathcal{P}_{(12)}+q^6\mathcal{P}_{(20)}-q^{10}\mathcal{P}_{(02)}+q^{12}\mathcal{P}_{(10)}+q^{16}\mathcal{P}_{(00)} \tag{28}$$

$$\begin{aligned}\check{R}_q^{\rm adj}(x)=\;&\Lambda_{(04)}(x,q)\mathcal{P}_{(04)}+\Lambda_{(12)}(x,q)\mathcal{P}_{(12)}+\Lambda_{(20)}(x,q)\mathcal{P}_{(20)}\\
&+\Lambda_{(02)}(x,q)\mathcal{P}_{(02)}+\Lambda_{(10)}(x,q)\mathcal{P}_{(10)}+\Lambda_{(00)}(x,q)\mathcal{P}_{(00)}\end{aligned} \tag{29}$$

$$\check{R}_q^{\rm adj}(0)=\check{R}_q^{\rm adj} \tag{30}$$

where, as usual, the project operators are the product of two quantum Clebsch–Gordan matrices

$$\mathcal{P}_N=\big(C_q^{(02)(02)}\big)_N\big(\tilde{C}_q^{(02)(02)}\big)_N\,. \tag{31}$$

Now, we are going to compute the coefficients $\Lambda_N(x,q)$

$$\check{R}_q^{\rm adj}(x)|N,N\rangle=\Lambda_N(x,q)|N,N\rangle\,. \tag{32}$$

In the computation, we need the quantum Clebsch–Gordan coefficients to combine the states $|m_1,m_2,m_3,m_4\rangle\equiv|m_1\rangle|m_2\rangle|m_3\rangle|m_4\rangle$ in the space $V_{(10)}\otimes V_{(10)}\otimes V_{(10)}\otimes V_{(10)}$ into the state $|N,N\rangle$.

Firstly, through the standard calculation, we obtain the quantum Clebsch–Gordan coefficients for the adjoint representation of U_qB_2. Denote by $|(0\,2), m\rangle$ the states in the adjoint representation, and by $|m_1m_2\rangle \equiv |m_1\rangle|m_2\rangle$ the states in the space $V_{(1\,0)} \otimes V_{(1\,0)}$, where the states is described by the enumerations given in figure 1. Owing to the symmetry of the quantum Clebsch–Gordan coefficients

$$\begin{aligned} |(0\,2), m\rangle &= \sum_{m_1m_2} \left(C_q\right)_{m_1m_2(0\,2)m} |m_1m_2\rangle \\ \left(C_q\right)_{m_1m_2(0\,2)m} &= -\left(C_{q^{-1}}\right)_{m_2m_1(0\,2)m} = -\left(C_{q^{-1}}\right)_{\bar{m}_1\bar{m}_2(0\,2)\bar{m}} \end{aligned} \tag{33}$$

we only need to list the following Clebsch–Gordan coefficients:

$$\begin{aligned} |(0\,2), 4\rangle &= ([2]/[4])^{1/2}\left\{q^{-1}|2\,1\rangle - q|1\,2\rangle\right\} \\ |(0\,2), 3\rangle &= [2]^{-1/2}f_2|(0\,2), 4\rangle = ([2]/[4])^{1/2}\left\{q^{-1}|2\,0\rangle - q|0\,2\rangle\right\} \\ |(0\,2), 2\rangle &= [2]^{-1/2}f_2|(0\,2), 3\rangle = ([2]/[4])^{1/2}\left\{q^{-1}|2\,\bar{1}\rangle - q|\bar{1}\,2\rangle\right\} \\ |(0\,2), 1\rangle &= f_1|(0\,2), 3\rangle = ([2]/[4])^{1/2}\left\{q^{-1}|1\,0\rangle - q|0\,1\rangle\right\} \\ |(0\,2), 0\rangle &= [2]^{-1/2}f_2|(0\,2), 1\rangle = ([2]/[4])^{1/2}\left\{|1\bar{1}\rangle + (q^{-1} - q)|0\,0\rangle - |\bar{1}\,1\rangle\right\} \\ |(0\,2), 0'\rangle &= ([3][2]/[6])^{1/2}\left\{f_1|(0\,2), 2\rangle - |(0\,2), 0\rangle\right\} \\ &= [2]\,([3]/[6][4])^{1/2}\left\{|2\,\bar{2}\rangle + (q^{-2} - 1)|1\,\bar{1}\rangle\right. \\ &\qquad \left. -(q^{-1} - q)|0\,0\rangle + (1 - q^2)|\bar{1}\,1\rangle - |\bar{2}\,2\rangle\right\}. \end{aligned} \tag{34}$$

From (34) we are able to compute the expansive expressions for the highest weight states in the Clebsch–Gordan series (27):

$$\begin{aligned} |(0\,4), (0\,4)\rangle &= |(0\,2), 4\rangle|(0\,2), 4\rangle \\ &= ([2]/[4])\left\{q^{-2}|2121\rangle - |2112\rangle - |1221\rangle + q^2|1212\rangle\right\} \end{aligned} \tag{35a}$$

$$\begin{aligned} |(12), (12)\rangle &= ([2]/[4])^{1/2}\left\{q^{-1}|(0\,2), 4\rangle|(0\,2), 3\rangle - q|(0\,2), 3\rangle|(0\,2), 4\rangle\right\} \\ &= ([2]/[4])^{3/2}\left\{q^{-3}|2120\rangle - q^{-1}|2102\rangle - q^{-1}|1220\rangle\right. \\ &\qquad \left. + q|1202\rangle - q^{-1}|2021\rangle + q|0221\rangle + q|2012\rangle - q^3|0212\rangle\right\} \end{aligned} \tag{35b}$$

where we see that the second-half terms of (35*a*) and (35*b*) can be obtained from the first-half terms by exchanging

$$F(q)|m_1m_2m_3m_4\rangle \longrightarrow \pm F(q^{-1})|m_4m_3m_2m_1\rangle \tag{36}$$

where the plus sign stands for (35*a*), and the minus sign for (35*b*). In the following we will use the abbreviated notation (S terms) (for (36) with plus sign) or (A terms) (minus sign) to replace the second-half terms, respectively. In this way (35*a*) and (35*b*) are rewritten as follows:

$$\begin{aligned} |(0\,4), (0\,4)\rangle &= ([2]/[4])\left\{q^{-2}|2121\rangle - \tfrac{1}{2}|2112\rangle - \tfrac{1}{2}|1221\rangle + (S \text{ terms})\right\} \\ |(1\,2), (1\,2)\rangle &= ([2]/[4])^{3/2}\left\{q^{-3}|2120\rangle - q^{-1}|2102\rangle - q^{-1}|1220\rangle + q|1202\rangle + (A \text{ terms})\right\}. \end{aligned}$$

In the same way we have

$$
\begin{aligned}
|(2\,0),(2\,0)\rangle &= ([3])^{-1/2}\left\{q^{-1}|(0\,2),4\rangle|(0\,2),2\rangle - |(0\,2),3\rangle|(0\,2),3\rangle + q|(0\,2),2\rangle|(0\,2),4\rangle\right\} \\
&= ([2]/[4])\,[3]^{-1/2}\left\{q^{-3}|212\bar{1}\rangle - q^{-1}|21\bar{1}2\rangle - q^{-1}|122\bar{1}\rangle\right. \\
&\qquad \left. + q|12\bar{1}2\rangle - q^{-2}|2020\rangle + \tfrac{1}{2}|2002\rangle + \tfrac{1}{2}|0220\rangle + (S\ \text{terms})\right\}
\end{aligned}
\tag{35c}
$$

$$
\begin{aligned}
|(0\,2),(0\,2)\rangle &= [3]^{-1}\left([6][5][2]/[10][4]\right)^{1/2}\left\{q^{-3}|(0\,2),4\rangle|(0\,2),0\rangle\right. \\
&\qquad - q^{-3}\left([3][2]/[6]\right)^{1/2}|(0\,2),4\rangle|(0\,2),0'\rangle - q^{-1}|(0\,2),3\rangle|(0\,2),1\rangle \\
&\qquad + q|(0\,2),1\rangle|(0\,2),3\rangle + q^{3}\left([3][2]/[6]\right)^{1/2}|(0\,2),0'\rangle|(0\,2),4\rangle \\
&\qquad \left. - q^{3}|(0\,2),0\rangle|(0\,2),4\rangle\right\} \\
&= \left([2]^2/[4]\right)\left([5][2]/[10][6][4]\right)^{1/2}\left\{-q^{-4}|212\bar{2}\rangle + q^{-2}|211\bar{1}\rangle\right. \\
&\qquad - q^{-6}|21\bar{1}1\rangle + q^{-4}|21\bar{2}2\rangle + q^{-2}|122\bar{2}\rangle - |121\bar{1}\rangle \\
&\qquad + q^{-4}|12\bar{1}1\rangle - q^{-2}|12\bar{2}2\rangle + (q^{-1}-q)\left([4]/[2]\right)\left(q^{-4}|2100\rangle\right. \\
&\qquad \left. - q^{-2}|1200\rangle\right) + \left([6]/[3][2]\right)\left(-q^{-3}|2010\rangle + q^{-1}|2001\rangle\right. \\
&\qquad \left.\left. + q^{-1}|0210\rangle - q|0201\rangle\right) + (A\ \text{terms})\right\}
\end{aligned}
\tag{35d}
$$

$$
\begin{aligned}
|(1\,0),(1\,0)\rangle &= \left([4]/[8][3]\right)^{1/2}\left\{q^{-3}|(0\,2),4\rangle|(0\,2),\bar{1}\rangle - q^{-2}|(0\,2),3\rangle|(0\,2),0\rangle\right. \\
&\qquad + q^{-1}|(0\,2),2\rangle|(0\,2),1\rangle + q|(0\,2),1\rangle|(0\,2),2\rangle \\
&\qquad \left. - q^{2}|(0\,2),0\rangle|(0\,2),3\rangle + q^{3}|(0\,2),\bar{1}\rangle|(0\,2),4\rangle\right\} \\
&\qquad = [2]\left([8][4][3]\right)^{-1/2}\left\{q^{-5}|210\bar{1}\rangle - q^{-3}|21\bar{1}0\rangle - q^{-3}|120\bar{1}\rangle\right. \\
&\qquad + q^{-1}|12\bar{1}0\rangle - q^{-3}|201\bar{1}\rangle - \left(q^{-4}-q^{-2}\right)|2000\rangle + q^{-3}|20\bar{1}1\rangle \\
&\qquad + q^{-1}|021\bar{1}\rangle + (q^{-2}-1)|0200\rangle - q^{-1}|02\bar{1}1\rangle + q^{-3}|2\bar{1}10\rangle \\
&\qquad \left. - q^{-1}|2\bar{1}01\rangle - q^{-1}|\bar{1}210\rangle + q|\bar{1}201\rangle\big) + (S\ \text{terms})\right\}
\end{aligned}
\tag{35e}
$$

$$
\begin{aligned}
|(0\,0),(0\,0)\rangle &= \left([4]/[8][5]\right)^{1/2}\left\{q^{-4}|(0\,2),4\rangle|(0\,2),\bar{4}\rangle - q^{-3}|(0\,2),3\rangle|(0\,2),\bar{3}\rangle\right. \\
&\qquad + q^{-2}|(0\,2),2\rangle|(0\,2),\bar{2}\rangle + q^{-1}|(0\,2),1\rangle|(0\,2),\bar{1}\rangle \\
&\qquad - |(0\,2),0\rangle|(0\,2),0\rangle - |(0\,2),0'\rangle|(0\,2),0'\rangle \\
&\qquad + q|(0\,2),\bar{1}\rangle|(0\,2),1\rangle + q^{2}|(0\,2),\bar{2}\rangle|(0\,2),2\rangle \\
&\qquad \left. - q^{3}|(0\,2),\bar{3}\rangle|(0\,2),3\rangle + q^{4}|(0\,2),\bar{4}\rangle|(0\,2),4\rangle\right\} \\
&= [2]\left([8][5][4]\right)^{-1/2}\left\{q^{-6}|21\bar{1}\bar{2}\rangle - q^{-4}|21\bar{2}\bar{1}\rangle - q^{-4}|12\bar{1}\bar{2}\rangle\right. \\
&\qquad + q^{-2}|12\bar{2}\bar{1}\rangle - q^{-5}|200\bar{2}\rangle + q^{-3}|20\bar{2}0\rangle + q^{-3}|020\bar{2}\rangle \\
&\qquad - q^{-1}|02\bar{2}0\rangle + q^{-4}|2\bar{1}1\bar{2}\rangle - q^{-2}|2\bar{1}\bar{2}1\rangle - q^{-2}|\bar{1}21\bar{2}\rangle \\
&\qquad + |\bar{1}2\bar{2}1\rangle + q^{-3}|100\bar{1}\rangle - q^{-1}|10\bar{1}0\rangle - q^{-1}|010\bar{1}\rangle \\
&\qquad + q|01\bar{1}0\rangle + \left([3][2]/2[6]\right)\left(-2|2\bar{2}2\bar{2}\rangle - 2\left(q^{-4}-q^{-2}+q^{2}\right)|1\bar{1}1\bar{1}\rangle\right. \\
&\qquad \left. + |1\bar{1}\bar{1}1\rangle + |\bar{1}11\bar{1}\rangle + |2\bar{2}\bar{2}2\rangle + |\bar{2}22\bar{2}\rangle\right) \\
&\qquad + \left(q^{-1}-q\right)\left(|2\bar{2}00\rangle + |002\bar{2}\rangle - q^{-1}|2\bar{2}1\bar{1}\rangle - q^{-1}|1\bar{1}2\bar{2}\rangle\right. \\
&\qquad \left. - q^{2}|1\bar{1}00\rangle - q^{2}|001\bar{1}\rangle - q|2\bar{2}\bar{1}1\rangle - q|\bar{1}12\bar{2}\rangle\right) \\
&\qquad \left. - \left(q^{-1}-q\right)^{2}\left([4]/2[2]\right)|0000\rangle + (S\ \text{terms})\right\}
\end{aligned}
\tag{35f}
$$

Now, substituting (22), (24) and (35) into (32), we obtain $\Lambda_N(x,q)$, and then, the solution $\check{R}_q^{\text{adj}}(x)$ of the Yang–Baxter equation related to the adjoint representation of $U_q B_2$ as follows:

$$\begin{aligned}
\check{R}_q^{\text{adj}}(x) = (q^4-x)(1-x)(1+xq^{10})(1+xq^{14}) \{ &(1-xq^4)(1+xq^6)(1-xq^8)(1+xq^{10})\mathcal{P}_{(04)} \\
&+ (x-q^4)(1+xq^6)(1-xq^8)(1+xq^{10})\mathcal{P}_{(12)} \\
&+ (1-xq^4)(x+q^6)(1-xq^8)(1+xq^{10})\mathcal{P}_{(20)} \\
&+ (x-q^4)(x+q^6)(1-xq^8)(1+xq^{10})\mathcal{P}_{(02)} \\
&+ (x-q^4)(1+xq^6)(x-q^8)(1+xq^{10})\mathcal{P}_{(10)} \\
&+ (1-xq^4)(x+q^6)(1-xq^8)(x+q^{10})\mathcal{P}_{(00)} \}
\end{aligned} \tag{37}$$

where the common factor $(q^4-x)(1-x)(1+xq^{10})(1+xq^{14})$ can be removed. In principle, this method can be generalized to the solutions of the Yang–Baxter equation related to the adjoint representations of $U_q B_\ell$, $U_q C_\ell$ and $U_q D_\ell$.

6. Rational solution for the adjoint representation

Through a standard limit process [1] we obtain the corresponding rational solution $R^{\text{adj}}(u/\eta)$ for the adjoint representation of $U_q B_2$:

$$\begin{aligned}
R^{\text{adj}}(u/\eta) &= \lim_{q\to 1} \frac{P\check{R}_q^{\text{adj}}(q^{2u/\eta})}{\left(1-q^{2u/\eta}\right)^2} \\
&= 4\left\{(1+2u/\eta)(1+4u/\eta)\left(P_{(04)}+P_{(20)}+P_{(00)}\right)\right. \\
&\quad \left. + (1-2u/\eta)(1+4u/\eta)\left(P_{(12)}+P_{(02)}\right) + (1-2u/\eta)(1-4u/\eta)P_{(10)}\right\}
\end{aligned} \tag{38}$$

where P is the transposition operator, and

$$P_N = \lim_{q\to 1} \mathcal{P}_N .$$

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References

[1] Ma Zhong-Qi 1993 *Yang–Baxter Equation and Quantum Enveloping Algebras* (Singapore: World Scientific)
[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Baxter R J 1972 *Ann. Phys.* **70** 193
[3] Jimbo M 1986 *Commun. Math. Phys.* **102** 537
[4] Ma Zhong-Qi 1991 *Commun. Theor. Phys.* **15** 37
[5] Jones V F R 1989 *Commun. Math. Phys.* **125** 459
Cheng Y, Ge M L and Xue K 1991 *Commun. Math. Phys.* **136** 195
[6] Jimbo M 1989 Introduction to the Yang–Baxter Equation *Braid Group, Knot Theory and Statistical Mechanics* ed C N Yang and M L Ge (Singapore: World Scientific) p 111
[7] Kulish P P, Reshetikhin N Yu and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393
Cherednik I V 1986 *Func. Anal. Appl.* **20** 87